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General-relativistic Radiative Transfer Theory in Refractive and Dispersive Media* **

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Summary. The transfer of radiation in an isotropic, refractive and dispersive medium in a curved spacetime is studied by applying the general-relativistic kinetic theory.

Relativistic geometric optics in a dispersive medium is first analyzed; we study in particular the change of frequency along the ray and the change of the direction of propagation as observed in the local rest frame of the medium. The general-relativistic kinetic theory (as formulated, e.g., by Ehlers) is then shown to be extendable to photons in a dispersive medium. Assuming that the radiation only undergoes continuous refraction and dispersion, the law of change of specific intensity along the ray can be obtained easily. The transfer of black-body radiation, the propagation of radiation in a cold plasma in the Friedmann universe, and the derivation of the explicit form of the transfer equation in a spherically symmetric, non-stationary dispersive medium (the generalization of Lindquist's equation) are discussed as examples.

Some astrophysical situations in which both refractive and dispersive effects of a medium, and the effects of the gravitational field on the radiation, can be important, are indicated.

Key words: radiative transfer — dispersive medium — general relativity

I. Introduction

Consider a plasma cloud in the vicinity of a black hole and electromagnetic waves propagating through the cloud. The waves can be generated by radiating matter, placed near the event horizon inside the cloud or by the primary in a double system the secondary of which is a black hole or, alternatively, by a distant quasar emitting radio waves which are then focussed by a large black hole surrounded by an accreting plasma in the nucleus of a galaxy. In all such cases refractive and dispersive effects of matter and, simultaneously, the effects of the curvature of spacetime due to the black hole may be important. Furthermore, as demonstrated recently (Virtaho and Jauho, 1973; see also Lerche, 1974), the refraction of radio rays in a pulsar magnetosphere may have remarkable consequences on the beaming of the radiation. For neutron stars with radii as small as ~ 1.6 times the gravitational radius, the curvature can have important effects not only on the structure of the star, but also on the radiation in the magnetosphere (cf. Cohen and Rosenblum, 1973).

The purpose of this paper is to take the first steps in a rigorous treatment of the problems indicated above, provided the geometric optics approximation is applicable. We consider the propagation of radiation in a general, isotropic, refractive and dispersive medium (not necessarily a plasma) in curved spacetime. The medium is assumed to be described by the index of refraction and by the four-velocity field.

The only extant comprehensive survey of general-relativistic geometric optics in an isotropic dispersive medium is given in the last chapter of Synge's book (Synge, 1960). It is based on the elegant abstract Hamiltonian theory of rays and waves without, however,

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** We learnt from one of the referees of our paper that recently E. Spiegel and J.L. Anderson independently tackled the problem of general-relativistic radiative transfer in dispersive media. Their results will appear in the Astrophysical Journal.

*** This work was started when the author was a student at the Department of Theoretical Physics of the Charles University. Some of the results included in Sections II and III were contained in the paper (unpublished) with which he won the First Prize in the Czechoslovak Competition for the Best Student Work in Theoretical Physics in June 1973.

being much concerned with the physical interpretation of the formalism. In Section 2, after summarizing the basic results of Synge in a slightly more physical way, we analyze in detail the photon propagation from the point of view of observers at rest in a dispersive medium. Our expressions for the change of frequency along a ray and of the change of direction of propagation, as seen by comoving observers, extend those given, for example, by Novikov and Thorne (1973) for non-refractive and non-dispersive media in general relativity; furthermore, they generalize the classical results (see, e.g., Pomraning, 1973) for a dispersive medium at rest in an inertial frame of special relativity. [It appears that *moving* dispersive media, in general with expansion, shear and rotation, have not been treated in this respect even in special relativity, except for the very recent papers by Lerche (1974), who considers plane, differentially sheared media.]

Since there exists a 4-dimensional Hamiltonian description of rays in dispersive media, it is possible to study "phenomenological" photons in such media within the rigorous framework of general-relativistic kinetic theory as summarized in reviews by Ehlers (1971, 1973) and by others. Section III is devoted to the extension of kinetic theory to such photons. In particular, we obtain the relation according to which the specific intensity changes along the ray, provided the radiation (except for known relativistic effects) undergoes continuous refraction and dispersion in a medium; we then write the transfer equation, when other interactions with the medium also take place.

[Specializing to a dispersive medium at rest in flat spacetime, we easily find the explicit form of the "streaming terms" in the transfer equation which, for a general space-, time- and frequency-dependent refractive index appear to have been first derived only by Pomraning (1973) with the use of a somewhat lengthy vector calculus.]

Three illustrative examples are given in Section IV: the propagation of black-body radiation in a transparent dispersive medium in curved spacetime; the transfer of radiation in a dispersive medium which is at rest with respect to the fundamental observers in the Friedmann universe; and the explicit form of the transfer equation in a spherically symmetric non-stationary (e.g., collapsing) dispersive medium are derived thus generalizing the equation of Lindquist (1966).

A few concluding remarks are added in the last section.

II. Relativistic Geometric Optics in a Dispersive Medium

A rigorous transition to relativistic geometric optics from the covariantly generalized Maxwell equations was studied systematically by Ehlers (1967). Ehlers considered non-dispersive isotropic media, but, on physical grounds, it is plausible that relativistic geometric optics should also represent an appropriate approximation in the case of dispersive (and even anisotropic) media whenever the following conditions hold in the instanteneous local Lorentz rest frame of the medium for an arbitrary event in spacetime[1]):

a) the typical wavelength λ of the waves is short compared with the distances over which the properties of the medium (the index of refraction, the velocity) vary;

b) λ is short compared with the distances over which the characteristics of the waves (the amplitude, wavelength, polarization) vary, so that the waves are locally monochromatic;

c) λ is short compared with the characteristic radius of curvature of the spacetime;

d) the properties of the medium change negligibly over one period of a typical wave.

Under these conditions the waves can be described by a rapidly changing, scalar function of position in spacetime – the phase $\Phi(x^\mu)$, so that a hypersurface of constant phase, $\Phi=\text{const}$, is associated with each wave; multiplying by the reduced Planck constant, we can equivalently use the function $f(x^\mu)=\hbar\Phi(x^\mu)$. Correspondingly, two vectors normal to the waves, can be introduced – the wave-vector

$$\boldsymbol{k}=\nabla\Phi\,, \tag{2.1}$$

and the 4-momentum vector

$$\boldsymbol{p}=\nabla f=\hbar\boldsymbol{k}\,. \tag{2.2}$$

[Synge (1960) does not introduce the vector $\boldsymbol{k}$ and calls the vector $\boldsymbol{p}$ the "frequency 4-vector". The "4-momentum vector" is used here because of identifying $\boldsymbol{p}$ with the 4-momentum of a "phenomenological" photon – see below.]

Assume the motion of the medium to be described by a unit timelike vector field $\boldsymbol{U}$. An observer at rest in the medium observes the frequency of the waves

$$\nu=-h^{-1}\boldsymbol{p}\cdot\boldsymbol{U}=-(2\pi)^{-1}\boldsymbol{k}\cdot\boldsymbol{U}=-(2\pi)^{-1}\nabla\Phi\cdot\boldsymbol{U}\,. \tag{2.3}$$

(Henceforth, $\boldsymbol{A}\cdot\boldsymbol{B}\equiv g_{\mu\nu}A^\mu B^\nu$ denotes the scalar product of 4-vectors $\boldsymbol{A}$, $\boldsymbol{B}$.) Let the observer carry an orthonormal triad $\{\boldsymbol{e}_a\}$ $(a=1,2,3)$ of spacelike vectors, orthogonal to his worldline, i.e. to the 4-velocity $\boldsymbol{U}$ at each event on the worldline. In the basis of the orthonormal tetrad $\{\boldsymbol{e}_a, \boldsymbol{U}\}$,

$$U^\alpha=\delta^\alpha_0\,,\qquad \nu=h^{-1}p^0=(2\pi)^{-1}k^0\,. \tag{2.4}$$

[1]) For a physical description of the transition to geometric optics in vacuum in curved spacetime, see also Misner *et al.* (1973). In case of a general dispersive medium, a rigorous transition may not be easy because the "optical metric" [see Eq. (2.21) in the following], used to advantage by Ehlers, loses its significance. Electromagnetic waves propagating in plasma in a curved spacetime were studied in the eikonal approximation recently by Madore (1974). In nonrelativistic physics the applicability of the geometric optics approximation was analyzed from a broad viewpoint by Weinberg (1962). [See also Pomraning (1973) and references therein.]

Following the terminology of Novikov and Thorne (1973) the frame of reference just described will be called the local rest frame (LRF) of the medium. LRF is, in general, an accelerated frame. At an arbitrary fixed event we can also consider a freely falling observer, with a local Lorentz frame (LLRF), with respect to which the medium is momentarily at rest. [It is in the LLRF that the metric is not only Minkowskian, but also all the connection coefficients vanish. Of course, at the event in question, (2.4) holds in the LLRF, too.]

An arbitrary observer (not necessarily at rest in the medium), having the 4-velocity $\bar{\boldsymbol{U}}$, sees the frequency of the waves to be

$$\bar{\nu}=-h^{-1}\boldsymbol{p}\cdot\bar{\boldsymbol{U}}=-(2\pi)^{-1}\boldsymbol{k}\cdot\bar{\boldsymbol{U}}=-(2\pi)^{-1}\nabla\Phi\cdot\bar{\boldsymbol{U}}\,.$$

The phase-velocity of the waves relative to an observer is defined as the minimum value of the velocities of all fictitious particles, riding on the waves, i.e. the particles with world-lines lying on the hypersurface $\Phi(x^\mu)=\text{const.}$ As measured by an observer with 4-velocity $\bar{\boldsymbol{U}}$ it is given by the invariant formula (Synge, 1960)

$$\frac{1}{\bar{v}_{ph}^2}=1+\frac{\boldsymbol{p}\cdot\boldsymbol{p}}{(\boldsymbol{p}\cdot\bar{\boldsymbol{U}})^2}\,, \tag{2.4}$$

which suggests a way of building up relativistic geometric optics in an isotropic medium. As in classical physics, we describe the medium by the index of refraction n which is the reciprocal value of the phase-velocity of the waves with respect to the medium. The refractive index will be considered to be a given scalar function of position in spacetime and of the frequency ν measured in the LRF or, equivalently, of either the invariant $\boldsymbol{p}\cdot\boldsymbol{U}=-h\nu$, or $\boldsymbol{k}\cdot\boldsymbol{U}=-2\pi\nu\equiv-\omega$ (ω is the "angular" frequency). Thus, the general-relativistic geometric optics in an isotropic dispersive medium can be based on the medium-equation

$$n^2=1+\frac{\boldsymbol{p}\cdot\boldsymbol{p}}{(\boldsymbol{p}\cdot\boldsymbol{U})^2}=1+\frac{\boldsymbol{k}\cdot\boldsymbol{k}}{(\boldsymbol{k}\cdot\boldsymbol{U})^2}\,, \tag{2.5}$$

in which $n=n(x,\ \boldsymbol{p}\cdot\boldsymbol{U}(x))$ and the metric $\boldsymbol{g}=\boldsymbol{g}(x)$ are assumed to be given. In the LRF the medium-equation reduces to the standard relation

$$n(x,\omega)=\frac{\ell}{\omega}\,, \tag{2.6}$$

ℓ being the magnitude of the ordinary wave 3-vector – the "spatial" part of $\boldsymbol{k}$ in LRF. Covariantly,

$$\ell^2=\boldsymbol{k}_\perp\cdot\boldsymbol{k}_\perp\,,\quad\text{where}\quad \boldsymbol{k}_\perp=\boldsymbol{h}\cdot\boldsymbol{k} \tag{2.7}$$

is the spacelike vector given by the contraction of $\boldsymbol{k}$ with the projection tensor into the 3-space orthogonal to $\boldsymbol{U}$ (the 3-space of the LRF);

$$\boldsymbol{h}=\boldsymbol{g}+\boldsymbol{U}\otimes\boldsymbol{U}\,. \tag{2.8}$$

The medium-equation can be solved for ω to yield a dispersion relation

$$\omega=\omega(\ell,x)\,. \tag{2.9}$$

In order to be able to apply the Hamiltonian description of the waves (see Synge, 1960 for details), we rewrite the medium-equation in the form

$$\mathscr{H}(x^\mu,p_\nu)=0\,, \tag{2.10}$$

where

$$\mathscr{H}(x^\mu,p_\nu)=\tfrac{1}{2}\left[g^{\mu\nu}p_\mu p_\nu-(n^2-1)\,(p_\sigma U^\sigma)^2\right]. \tag{2.11}$$

Note that the *covariant* components of $\boldsymbol{p}$ are considered as fundamental in $\mathscr{H}$. Since $p_\mu=\hbar\partial\Phi/\partial x^\mu$, Eq. (2.10) is a partial differential equation for the phase. The characteristic curves of this equation, called *the rays*, obey the Hamiltonian equations

$$\frac{dx^\mu}{dw}=\frac{\partial\mathscr{H}}{\partial p_\mu}\,, \tag{2.12a}$$

$$\frac{dp_\mu}{dw}=-\frac{\partial\mathscr{H}}{\partial x^\mu}\,, \tag{2.12b}$$

where w is a parameter. Starting from an arbitrary event x with an initial momentum vector $\boldsymbol{p}$ [restricted only by (2.5) at x], we can determine the ray $x(w)$ and the momentum vector along the ray by solving the ordinary differential Eqs. (2.12). Since

$$\frac{\partial\mathscr{H}}{\partial p_\mu}=p^\mu-(n^2-1)\,(\boldsymbol{p}\cdot\boldsymbol{U})U^\mu+h^{-1}n\frac{\partial n}{\partial\nu}(\boldsymbol{p}\cdot\boldsymbol{U})^2U^\mu\,, \tag{2.13}$$

the space-time direction of the tangent vector to the ray,

$$\frac{d}{dw}=\frac{dx^\mu}{dw}\frac{\partial}{\partial x^\mu}=\frac{\partial\mathscr{H}}{\partial p_\mu}\frac{\partial}{\partial x^\mu}\,, \tag{2.14}$$

does not, in general, coincide with that of $\boldsymbol{p}$, however, it lies in the 2-surface spanned by $\boldsymbol{p}$ and $\boldsymbol{U}$. In particular, in LRF the spatial direction of $\frac{d}{dw}$ is the same as of $\boldsymbol{p}$; moreover, it is seen from (2.13) that

$$\boldsymbol{h}\cdot\frac{d}{dw}=\boldsymbol{h}\cdot\boldsymbol{p}\,. \tag{2.15}$$

Since signals (information) are carried along the rays, the vector $\frac{d}{dw}$ must be non-spacelike. Normalizing it (if it is time-like) and projecting onto $\bar{\boldsymbol{U}}$, we obtain $(1-\bar{v}_{\text{ray}}^2)^{-1/2}$, with $\bar{v}_{\text{ray}}$ being the magnitude of the velocity of the ray, as observed by an observer having the 4-velocity $\bar{\boldsymbol{U}}$. The expression for $\bar{v}_{\text{ray}}^2$,

$$\bar{v}_{\text{ray}}^2=1+g_{\mu\nu}\frac{\partial\mathscr{H}}{\partial p_\mu}\frac{\partial\mathscr{H}}{\partial p_\nu}\left(\bar{U}_\sigma\frac{\partial\mathscr{H}}{\partial p_\sigma}\right)^{-2}\,, \tag{2.16}$$

actually holds for an arbitrary character of the vector $\frac{d}{dw}$, but $\bar{v}_{\text{ray}}\leqq 1$ only if $\frac{d}{dw}$ is non-spacelike, i.e. if

$g_{\mu\nu}\frac{\partial\mathscr{H}}{\partial p_\mu}\frac{\partial\mathscr{H}}{\partial p_\nu}\leqq 0$. Specializing to the LRF, we find from (2.15) [using (2.13) and (2.6)] that the ray-velocity is the group-velocity as known in classical physics[2]):

$$v_{\text{ray}}=\left[\frac{\partial}{\partial\nu}(n\nu)\right]^{-1}=\frac{\partial\omega}{\partial k}=v_{gr}\,. \tag{2.17}$$

In particular, for waves with the frequency ν propagating in cold plasma with the plasma frequency $\nu_p<\nu$ (see e.g. Bekefi, 1966), the refractive index is given by

$$n=(1-\nu_p^2/\nu^2)^{1/2}\,, \tag{2.18}$$

so that

$$v_{ph}=n^{-1}=(1-\nu_p^2/\nu^2)^{-1/2}\,, \tag{2.19}$$

and

$$v_{gr}=(1-\nu_p^2/\nu^2)^{1/2}=n\,. \tag{2.20}$$

What is the photon in the described formalism? Synge writes: "In view of $\nu=-h^{-1}p_\mu U^\mu$, it seems appropriate to take the frequency vector p_μ to be the 4-momentum of a photon associated with a system of waves, and the history of a photon to be a ray." Although this view will be adopted in the following – we have called $\boldsymbol{p}$ the momentum vector from the very beginning – it should be mentioned that this interpretation is not unique, if we want the 4-momentum of the "phenomenological" photon to be derived from the quantum theory of the electromagnetic field in a medium. To start with, such a theory uses a classical expression for the energy-momentum tensor of the field in the medium, but (despite much discussion of this topic over many years) no such unique expression seems to be generally accepted. Muzikář (1956), among others, constructed the Lorentz – covariant phenomenological quantum theory of the electromagnetic field in a medium with a constant refractive index and, indeed, he obtained $\boldsymbol{p}=\hbar\boldsymbol{k}$ for the 4-momentum of the photon, having used the Minkowski or the canonical energy-momentum tensor. However, for the Abraham tensor, for example, Muzikář obtained the 4-momentum of a photon proportional to the tangent vector to the ray. In this paper we adhere to the relation $\boldsymbol{p}=\hbar\boldsymbol{k}$ not only because the Minkowski tensor seems to be the most natural [see, e.g., Schmutzer (1968), Møller (1972) and in particular, the most recent paper by Israel (1975) for some concrete arguments in favour of the Minkowski tensor] but, primarily, because if $\boldsymbol{p}=\hbar\boldsymbol{k}$, then our photons are identical with the photons (or "light particles") associated with wave packets in media (plasmas in particular) at rest in an inertial frame in the standard literature [see, e.g., Bekefi (1967), Chapter I or Pomraning (1973), Chapter V]. (Notice, moreover, that we could base the kinetic theory on the "$\boldsymbol{k}$-space" in any case and the "$\boldsymbol{k}$-space" is trivially related to the "$\boldsymbol{p}$-space" provided that $\boldsymbol{p}=\hbar\boldsymbol{k}$.)

In contrast to the photon in vacuum, in a medium $\boldsymbol{p}$ is neither tangent to the world-line of the photon, nor parallel-transported along it. It is well-known, however, that for *non*-dispersive media one can introduce the "optical metric"

$$\tilde{\boldsymbol{g}}=\boldsymbol{g}+(1-n^{-2})\boldsymbol{U}\otimes\boldsymbol{U}\,, \tag{2.21}$$

with respect to which the photon-in-medium has the same properties as the photon-in-vacuum: Eq. (2.10) becomes

$$\tfrac{1}{2}\tilde{g}^{\mu\nu}p_\mu p_\nu=0\,, \tag{2.22}$$

and the ray equations have the form

$$\frac{dx^\mu}{dw}=\tilde{p}^\mu\equiv\tilde{g}^{\mu\nu}p_\nu\,, \tag{2.23a}$$

$$\frac{dp_\mu}{dw}=-\tfrac{1}{2}\tilde{g}^{\iota\varkappa}{}_{,\mu}p_\iota p_\varkappa\,, \tag{2.23b}$$

so that the rays are null geodesics with respect to the optical metric.

Let us now analyze the propagation of a photon from the point of view of observers at rest in the medium. [See Ellis (1971) and Novikov and Thorne (1973) for discussions of the photon propagation with respect to observers at rest in a medium without refractive and dispersive properties.] At each event along the ray we can decompose both the tangent vector to the ray and the 4-momentum vector into their projections on $\boldsymbol{U}$ and into the 3-space orthogonal to $\boldsymbol{U}$. Regarding (2.16) (with $\bar{\boldsymbol{U}}=\boldsymbol{U}$), (2.13), (2.14) and (2.17), we find

$$\begin{aligned}\frac{d}{dw}&=\left(-U_\iota\frac{\partial\mathscr{H}}{\partial p_\iota}\right)[\boldsymbol{U}+v_{gr}\boldsymbol{N}]\\&=nh\nu v_{gr}^{-1}(\boldsymbol{U}+v_{gr}\boldsymbol{N})\,,\end{aligned} \tag{2.24}$$

and using (2.10) with (2.11), we arrive at

$$\boldsymbol{p}=(-\boldsymbol{p}\cdot\boldsymbol{U})\,[\boldsymbol{U}+n\boldsymbol{N}]=h\nu(\boldsymbol{U}+n\boldsymbol{N})\,. \tag{2.25}$$

Here $\boldsymbol{N}$ is the unit spacelike vector in the direction of the projection $\boldsymbol{h}\cdot\frac{d}{dw}$ [$=\boldsymbol{h}\cdot\boldsymbol{p}$ by (2.15)]. In the LRF, it is purely spatial ($\boldsymbol{N}\cdot\boldsymbol{U}=0$) and it determines the spatial direction of propagation of the ray. Since $dx^\mu=\frac{\partial\mathscr{H}}{\partial p_\mu}dw$, a small increment dw in the parameter along the ray will be seen in the LRF to correspond to an interval dt and a spatial distance dl, given as a consequence of (2.24), by the invariant expressions

$$dt=\left(-U_\iota\frac{\partial\mathscr{H}}{\partial p_\iota}\right)dw=nh\nu v_{gr}^{-1}dw\,, \tag{2.26}$$

$$dl=\left(-U_\iota\frac{\partial\mathscr{H}}{\partial p_\iota}\right)v_{gr}dw=nh\nu dw=v_{gr}dt\,. \tag{2.27}$$

[2]) Note the misprints in Synge (1960): in Eq. (32), p. 378, q^{-2} and q^{-1} should be replaced by q^2 and q, and below (33), the relation $v=q$ should read $v=q^{-1}$.

Using the projection tensor (2.8) we can also write a simple relation for the spatial projection of the tangent to the ray

$$h^{\mu}_{\nu}\frac{dx^{\nu}}{dl}=N^{\mu}\,, \tag{2.28}$$

which in the LRF goes over into the classical form

$dx^a/dl=N^a(a=1,2,3)$.

The first covariant derivatives of the 4-velocity vector of the medium can be decomposed at each event as follows (see, e.g., Ellis, 1971):

$$U_{\alpha;\beta}=\omega_{\alpha\beta}+\sigma_{\alpha\beta}+\tfrac{1}{3}\theta h_{\alpha\beta}-a_{\alpha}U_{\beta}\,, \tag{2.29}$$

where $a_{\alpha}\equiv U_{\alpha;\beta}U^{\beta}$ is the 4-acceleration of the medium, $\theta\equiv U^{\alpha}{}_{;\alpha}$ is the volume expansion of the medium, $\omega_{\alpha\beta}\equiv\frac{1}{2}(h^{\iota}_{\beta}U_{\alpha;\iota}-h^{\iota}_{\alpha}U_{\beta;\iota})=-\omega_{\beta\alpha}$ is the vorticity and $\sigma_{\alpha\beta}\equiv\frac{1}{2}(h^{\iota}_{\beta}U_{\alpha;\iota}+h^{\iota}_{\alpha}U_{\beta;\iota})-\frac{1}{3}\theta h_{\alpha\beta}=\sigma_{\beta\alpha}$ is the shear tensor of the medium. (Let us recall that $a_{\alpha}U^{\alpha}=\omega_{\alpha\beta}U^{\alpha}=\sigma_{\alpha\beta}U^{\alpha}=0$.)

The change in the frequency ν with respect to observers at rest in the medium occurring along the ray for an increment dw of the parameter, is determined by

$$d(p_{\alpha}U^{\alpha})=(p_{\alpha}U^{\alpha})_{;\beta}\frac{dx^{\beta}}{dw}dw$$

$$=\left(\frac{dp_{\alpha}}{dw}-\Gamma^{\iota}_{\alpha\beta}p_{\iota}\frac{dx^{\beta}}{dw}\right)U^{\alpha}dw+U_{\alpha;\beta}p^{\alpha}\frac{dx^{\beta}}{dw}dw\,.$$

Substituting for dx^{β}/dw and dp_{α}/dw from (2.12a) and (2.12b), where (2.13) and

$$\frac{\partial\mathscr{H}}{\partial x^{\alpha}}=\tfrac{1}{2}g^{\mu\nu}{}_{,\alpha}p_{\mu}p_{\nu}-nn_{,\alpha}(\boldsymbol{p}\cdot\boldsymbol{U})^2$$

$$+\left[h^{-1}n\frac{\partial n}{\partial\nu}(\boldsymbol{p}\cdot\boldsymbol{U})^2-(n^2-1)(\boldsymbol{p}\cdot\boldsymbol{U})\right]p_{\iota}U^{\iota}{}_{,\alpha} \tag{2.30}$$

are used, and decomposing $U_{\alpha;\beta}$ according to (2.29), we find

$$d(p_{\alpha}U^{\alpha})=(-\boldsymbol{p}\cdot\boldsymbol{U})\,[(\sigma_{\alpha\beta}N^{\alpha}N^{\beta}+\tfrac{1}{3}\theta)n+\boldsymbol{a}\cdot\boldsymbol{N}$$
$$+\nabla n\cdot\boldsymbol{U}]nh\nu dw\,, \tag{2.31}$$

or, in view of (2.27),

$$\frac{d\nu}{dl}=-[(\sigma_{\alpha\beta}N^{\alpha}N^{\beta}+\tfrac{1}{3}\theta)n+\boldsymbol{a}\cdot\boldsymbol{N}+\nabla n\cdot\boldsymbol{U}]\nu\,. \tag{2.32}$$

The first two terms, $-(\sigma_{\alpha\beta}N^{\alpha}N^{\beta}+\frac{1}{3}\theta)n\nu$, caused by the expansion of the refractive medium along the direction of propagation of the photon, represent the "Doppler" or "cosmological" part of the redshift (2.32). The third term, $-(\boldsymbol{a}\cdot\boldsymbol{N})\nu$, caused by the acceleration of the medium (i.e. of the LRF) is the "gravitational redshift" (it is the only term which does not depend on the refractive index). The fourth term, $-(\nabla n\cdot\boldsymbol{U})\nu$, caused by the explicit dependence of the refractive index on time in the LRF, may be called the "refractive redshift". If $n=1$, (2.32) reduces to the result given, e.g., by Ellis (1971), or Novikov and Thorne (1973); while for $\boldsymbol{U}=$const we obtain $\frac{d\nu}{dl}=-(\nabla n\cdot\boldsymbol{U})\nu$, which in the LLRF agrees with the relation $\frac{d\nu}{dl}=-\nu\frac{dn}{\partial t}$, known from classical physics (see Pomraning, 1973, Chapter V).

An intuitive derivation of (2.32) in the very simple case of a dispersive medium in the Friedmann universe will be given in Section 4.

Next, we wish to find out how the direction of propagation of a photon changes from the viewpoint of observers at rest in the medium. For this purpose, first calculate the absolute derivative of the 4-momentum vector along the ray, $p_{\beta;\gamma}\frac{dx^{\gamma}}{dw}=\frac{dp_{\beta}}{dw}-\Gamma^{\sigma}_{\beta\gamma}p_{\sigma}\frac{dx^{\gamma}}{dw}$; use (2.12a) and (2.12b) together with (2.13) and (2.30), and substitute the decomposition of $\boldsymbol{p}$ as given by (2.25) into the result obtained. In this way find [also considering (2.17)] that

$$p_{\beta;\gamma}\frac{dx^{\gamma}}{dw}=(nh\nu)^2[n^{-1}n_{,\beta}-(v_{gr}^{-1}-n^{-1})U_{\sigma;\beta}N^{\sigma}]\,. \tag{2.33}$$

Then calculate $p_{\beta;\gamma}\frac{dx^{\gamma}}{dw}$ starting out directly from (2.25) and (2.31). By comparing the result with (2.33), find an expression for $N_{\beta;\gamma}\frac{dx^{\gamma}}{dw}$. Project it into the LRF and, in accordance with (2.27), write $h^{\beta}_{\alpha}N_{\beta;\gamma}\frac{dx^{\gamma}}{dw}=nh\nu h^{\beta}_{\alpha}N_{\beta;\gamma}\times\frac{dx^{\gamma}}{dl}\equiv nh\nu h^{\beta}_{\alpha}\frac{DN_{\beta}}{dl}$. The described procedure yields

$$h^{\beta}_{\alpha}\frac{DN_{\beta}}{dl}=(h^{\beta}_{\alpha}-N_{\alpha}N^{\beta})\,[n^{-1}n_{,\beta}-v_{gr}^{-1}N^{\sigma}U_{\sigma;\beta}-(nv_{gr})^{-1}a_{\beta}]$$
$$+n^{-1}h^{\beta}_{\alpha}(U_{\sigma;\beta}-U_{\beta;\sigma})N^{\sigma}\,.$$

Using the decomposition (2.29) in this result one obtains

$$h^{\beta}_{\alpha}\frac{DN_{\beta}}{dl}=n^{-1}n_{,\beta}(h^{\beta}_{\alpha}-N_{\alpha}N^{\beta})$$
$$+(nv_{gr})^{-1}\{[n(N_{\alpha}N^{\gamma}-\delta^{\gamma}_{\alpha})\sigma_{\gamma\beta}+(n-2v_{gr})\omega_{\alpha\beta}]N^{\beta}$$
$$+(\boldsymbol{a}\cdot\boldsymbol{N})N_{\alpha}-a_{\alpha}\}\,. \tag{2.34}$$

Consider a medium at rest in a Lorentz frame ($\boldsymbol{U}=$const). In the usual Galilean coordinates in this frame, (2.34) reads

$$\frac{dN_a}{dl}=n^{-1}[n_{,a}-(N^bn_{,b})N_a]\qquad(a=1,2,3)$$

in agreement with the classical result [see Pomraning (1973), Chapter V; for a medium with the refractive index a function of position in space only, see also, e.g. Born and Wolf, (1964), Chapter 3).

The system of Eqs. (2.28), (2.32) and (2.34), together with relations (2.25), (2.26) and (2.27) imply the original Hamiltonian system (2.12a) and (2.12b) with (2.10) and (2.11).

III. Relativistic Kinetic Theory of Photons in a Dispersive Medium

Among recent extensive reviews on the kinetic theory of particles in curved spacetime, the most suitable one, for our purposes, is contained in the article by Ehlers (1973). Only there is a Hamiltonian description of the phase flow (in the cotangent bundle over spacetime) explicitely used. In this section we will adhere to the concepts and mathematical tools of the Ehlers review (also referring the reader to it in connection with the formalism of Cartan's differential forms).

The particles we wish to study move along the world-lines given by (2.12a) with 4-momenta determined by (2.12b). In general, considering x^α, p_α as fundamental variables, we may define the 8-dimensional one-particle phase space P as the collection of all possible instantaneous states of photons and of particles with non-zero rest mass, described by pairs (x^α, p_α). The phase space is thus a part of the cotangent bundle $T^*(M)$ over spacetime M. In the kinetic theory of particles with non-zero rest mass and of photons which are not affected by refraction or dispersion one defines P as the set of all non-spacelike future-directed covariant vectors at arbitrary events, so that the boundary of P in $T^*(M)$ consists of the states of particles with zero rest mass. Since the 4-momentum of a photon in a medium may be a spacelike vector (it is whenever $n>1$), we also have to include spacelike vectors into P. We only require the 4-momentum to be future-directed as measured in the LRF (i.e. the condition covariantly reading $\boldsymbol{p}\cdot\boldsymbol{U}\leqq 0$ at each event). The states of photons in a medium with refractive and dispersive properties do not form the boundary of the phase space of all particles in general but, similarly to the states of particles with a given rest mass, they form a 7-dimensional sub-phase space. It is given by Eqs. (2.10), (2.11) and, hereafter, will be denoted by the symbol P_n.

The paths of photons determine the phase flow in P, i.e. the congruence of curves with the tangent vector field – the Liouville vector (operator) –

$$\boldsymbol{L}=\frac{\partial\mathscr{H}}{\partial p_\alpha}\frac{\partial}{\partial x^\alpha}-\frac{\partial\mathscr{H}}{\partial x^\alpha}\frac{\partial}{\partial p_\alpha}, \tag{3.1}$$

where $\partial\mathscr{H}/\partial p_\alpha$ and $\partial\mathscr{H}/\partial x^\alpha$ are given by (2.13) and (2.30).

Since there is a simple correspondence, given by the metric, between covariant and contravariant vectors, we can use either (x^α, p_α), or (x^α, p^α) as local coordinates in P, in the latter case obtaining a representation of $\boldsymbol{L}$ in the form

$$\boldsymbol{L}=\frac{\partial\mathscr{H}}{\partial p_\alpha}\frac{\partial}{\partial x^\alpha}-\left[\frac{\partial\mathscr{H}}{\partial x^\sigma}+\frac{\partial\mathscr{H}}{\partial p_\varrho}g_{\sigma\tau,\varrho}p^\tau\right]g^{\sigma\alpha}\frac{\partial}{\partial p^\alpha}, \tag{3.2}$$

which, for photons in vacuum, reduces to the standard expression

$$\boldsymbol{L}=p^\alpha\frac{\partial}{\partial x^\alpha}-\Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma\frac{\partial}{\partial p^\alpha}.$$

Note that in the case of a non-dispersive medium the Liouville operator may also be represented analogously as in vacuum,

$$\boldsymbol{L}=\tilde{p}^\alpha\frac{\partial}{\partial x^\alpha}-\tilde{\Gamma}^\alpha_{\beta\gamma}\tilde{p}^\beta\tilde{p}^\gamma\frac{\partial}{\partial\tilde{p}^\alpha}, \tag{3.3}$$

if the optical metric (2.21) is employed. [$\tilde{\Gamma}^\alpha_{\beta\gamma}$ are the Christoffel symbols formed by means of (2.21).]

The Liouville vector is tangent to P_n: $\boldsymbol{L}(\mathscr{H})=\frac{\partial\mathscr{H}}{\partial p_\alpha}\frac{\partial\mathscr{H}}{\partial x^\alpha}-\frac{\partial\mathscr{H}}{\partial x^\alpha}\frac{\partial\mathscr{H}}{\partial p_\alpha}=0$, so that we may also use (3.1) in P_n and later take into account the dispersion relation $\mathscr{H}=0$, or, we may restrict $\boldsymbol{L}$ to P_n from the very beginning and write, for example,

$$\boldsymbol{L}_n=\frac{\partial\mathscr{H}}{\partial p_\alpha}\frac{\partial}{\partial x^\alpha}-\frac{\partial\mathscr{H}}{\partial x^i}\frac{\partial}{\partial p_i},\qquad (i=1,2,3)$$

assuming p_0 to be determined by the dispersion relation (2.10). (The root for which $\boldsymbol{p}\cdot\boldsymbol{U}\leqq 0$ must be chosen.)

Following Ehlers (1973), we now define the Riemannian volume element – a 4-form

$$\eta=\sqrt{-g}\,dx^{0123}=\frac{1}{4!}\eta_{\alpha\beta\gamma\delta}dx^{\alpha\beta\gamma\delta}, \tag{3.4}$$

where $g=\det(g_{\alpha\beta})$ and an abbreviation $dx^{\alpha\beta\gamma\delta}=dx^\alpha\wedge dx^\beta\wedge dx^\gamma\wedge dx^\delta$ for the wedge product is used (also, $dx^{\alpha\beta\gamma}\equiv dx^\alpha\wedge dx^\beta\wedge dx^\gamma$ etc.). Similarly, a volume element in 4-momentum space can be defined,

$$\pi=(\sqrt{-g})_x dp^{0123}=\frac{1}{4!}\eta_{\alpha\beta\gamma\delta}dp^{\alpha\beta\gamma\delta}. \tag{3.5}$$

The flux of a vector field $\boldsymbol{V}$ through a hypersurface $\mathscr{S}$ in spacetime is given by

$$\int_\mathscr{S}\boldsymbol{V}\cdot\eta=\int_\mathscr{S}V^\alpha\sigma_\alpha,$$

where $\boldsymbol{V}\cdot\eta$ is the contraction of η with $\boldsymbol{V}$ and a covariant 3-form

$$\sigma_\alpha=\frac{1}{3!}\eta_{\alpha\beta\gamma\delta}dx^{\beta\gamma\delta} \tag{3.6}$$

is the hypersurface element. If $x^0=\text{const}$ in the LRF is chosen, as a hypersurface $\mathscr{S}$,

$$\sigma_\alpha=U_\alpha dx^{123}, \tag{3.7}$$

with $U_\alpha = -\delta^0_\alpha$. Further, the canonical Lebesgue measure in phase space P can be introduced by

$$\Omega = -dx^{0123} \wedge dp_{0123} = \eta \wedge \pi\,. \tag{3.8}$$

The contraction of Ω with the Liouville vector $\boldsymbol{L}$ represented by (3.2) leads to the 7-form

$$\omega \equiv \boldsymbol{L} \cdot \Omega = \frac{\partial \mathscr{H}}{\partial p_\alpha} \sigma_\alpha \wedge \pi - \frac{1}{6} \eta_{\alpha\beta\gamma\delta} \left[\frac{\partial \mathscr{H}}{\partial x^\sigma} + \frac{\partial \mathscr{H}}{\partial p_\varrho} g_{\sigma\tau,\varrho} p^\tau\right] g^{\sigma\alpha} dp^{\beta\gamma\delta} \wedge \eta\,. \tag{3.9}$$

A hypersurface $\mathscr{S}$ in spacetime and 4-momenta regions at each point of $\mathscr{S}$ form a hypersurface Σ in P. When restricted to such a Σ, the 7-form ω is simply given by

$$\omega = \frac{\partial \mathscr{H}}{\partial p_\alpha} \sigma_\alpha \wedge \pi\,. \tag{3.10}$$

In particular, choosing a hypersurface $x^0 = \text{const}$ in the LRF as $\mathscr{S}$, we obtain

$$\omega = -\frac{\partial \mathscr{H}}{\partial p_0} dx^{123} \wedge dp^0 \wedge dp^{123}\,.$$

Defining a divergence operator relative to Ω by $d(\boldsymbol{W} \cdot \Omega) = (\mathrm{Div}\boldsymbol{W})\Omega$, where $\boldsymbol{W}$ is an arbitrary vector field on P, we see that $\mathrm{Div}\boldsymbol{L} = \frac{\partial^2 \mathscr{H}}{\partial x^\alpha \partial p_\alpha} - \frac{\partial^2 \mathscr{H}}{\partial p_\alpha \partial x^\alpha} = 0$. It follows (see Ehlers, 1973) that

$$\mathscr{L}_{\boldsymbol{L}} \Omega = d\omega = 0\,, \quad \mathscr{L}_{\boldsymbol{L}} \omega = 0\,, \tag{3.11}$$

where $\mathscr{L}_{\boldsymbol{L}}$ is the Lie derivative with respect to the phase flow. This invariance of Ω and ω with respect to the phase flow immediately implies Liouville's theorem: Ω and ω – measures of domains in P are unchanged along the phase flow generated by $\boldsymbol{L}$. Equations (3.1), (3.2) and, of course, (3.4)–(3.11) hold in the same form regardless of whether photons propagating in a medium in curved spacetime, or particles with a given arbitrary rest mass freely moving in curved spacetime, are considered; clearly, phenomenological photons in the medium can be incorporated into the framework of the general relativistic kinetic theory.

Let us now concentrate on the 7-dimensional phase space P_n and try to find analogues of Ω and ω from P. Using the fact that not only Ω, but also $\mathscr{F}(\mathscr{H})\Omega$, with $\mathscr{F}$ being an arbitrary function of $\mathscr{H}$, is an $\boldsymbol{L}$-invariant measure on P, we define an $\boldsymbol{L}$-invariant measure on P_n by

$$\Omega_n = 2F(\boldsymbol{p} \cdot \boldsymbol{U})\delta[2\mathscr{H}(x^\alpha, p_\alpha)]\Omega\,, \tag{3.12}$$

where $F = 1$ if $\boldsymbol{p} \cdot \boldsymbol{U} \leqq 0$, otherwise $F = 0$, and δ is the Dirac distribution function. Although (3.12) is manifestly covariant, it is convenient to integrate the δ-function over p^0 taking into account the dispersion relation (2.10), (2.11). In this way one finds

$$\Omega_n = \eta \wedge \pi_n\,, \tag{3.13}$$

where η is the space-time volume element (3.4) and

$$\pi_n = \frac{\sqrt{-g}\, dp^{123}}{\left|p_0 + \left[nh^{-1} \frac{\partial n}{\partial \nu} (\boldsymbol{p} \cdot \boldsymbol{U})^2 - (n^2 - 1)(\boldsymbol{p} \cdot \boldsymbol{U})\right] U_0\right|}\,. \tag{3.14}$$

In particular, in Galilean coordinates in the LLRF we obtain the simple expression

$$\pi_n = \frac{dp^{123}}{nh\nu \left|\nu \frac{\partial n}{\partial \nu} + n\right|} = \frac{v_{\mathrm{gr}} dp^{123}}{nh\nu}\,, \tag{3.15}$$

which for $n = 1$ reduces to the standard special relativistic result. Now, consider a hypersurface $\mathscr{S}$ in spacetime and, at each point of $\mathscr{S}$, photon 4-momenta constrained by (2.10). In this way a 6-dimensional hypersurface Σ_n in P_n is formed, the measure on it [the analogue of (3.10)] being given by

$$\omega_n = \frac{\partial \mathscr{H}}{\partial p_\alpha} \sigma_\alpha \wedge \pi_n\,. \tag{3.16}$$

Regarding (2.13), (3.7) and (3.15), we see that in the LLRF,

$$|\omega_n| = dx^{123} \wedge dp^{123}\,, \tag{3.17}$$

so that ω_n is an ordinary phase-space volume element. It is easily seen that (3.17) holds for an arbitrary observer with the 4-velocity $\bar{\boldsymbol{U}}$, provided he uses Galilean coordinates and his hypersurface $x^0 = \text{const}$, so that

$$\sigma_\alpha = \bar{U}_\alpha dx^{123}\,, \quad \bar{U}_\alpha = -\delta^0_\alpha\,.$$

Analogously as in P, we can prove Liouville's theorem

$$\mathscr{L}_{\boldsymbol{L}_n} \Omega_n = d\omega_n = 0\,, \quad \mathscr{L}_{\boldsymbol{L}_n} \omega_n = 0\,.$$

Following the rigorous arguments, given by Ehlers (1973) for photons in vacuum, one can introduce a unique non-negative distribution function f on P_n such that

$$N[\Sigma_n] = \int_{\Sigma_n} f \omega_n$$

determines the mean number of states of the photons in the medium, intersecting the hypersurface Σ_n. The distribution function is a scalar on P_n, which for an arbitrary local observer coincides with the ordinary distribution function owing to (3.17). If the photons stream through the dispersive medium without undergoing collisons, f satisfies Liouville's equation

$$\boldsymbol{L}(f) = 0\,. \tag{3.18}$$

(Recall that it does not matter whether we use $\boldsymbol{L}$ instead of $\boldsymbol{L}_n$, taking into account the dispersion relation later.) When, except for undergoing continuous refraction and dispersion, the radiation interacts with the medium so that there is spontaneous and stimulated emission of radiation by the medium, and the absorption and scattering, the distribution function satisfies the equation of radiative transfer (the Boltzmann kinetic equation)

$$\boldsymbol{L}(f) = g\,, \tag{3.19}$$

where the right-hand side describes a change of f due to the interactions mentioned above. In this paper we are primarily interested in incorporating refractive and dispersive effects into the "streaming terms" of the equation of radiative transfer – the left-hand side of (3.19) –; and the form of g will not be discussed. Since it is not changed by refraction and dispersion, we may just refer to the literature (see e.g. Lindquist, 1966; Ehlers, 1973; Novikov and Thorne, 1973).

In astrophysics one usually works with the specific radiation intensity I_ν, instead of the distribution function f. Hereafter, we confine the discussion to the LLRF. Introducing the direction angles θ, φ in the LLRF we can express the volume element π_n given by (3.15) in the form

$$\pi_n = \frac{v_{gr}}{nh\nu} p^2 dp \sin\theta d\theta d\varphi ,$$

where (using the dispersion relation)

$$p = [(p^1)^2 + (p^2)^2 + (p^3)^2]^{1/2} = nh\nu ,$$

so that

$$\pi_n = v_{gr} nh^2 \nu d(n\nu) \sin\theta d\theta d\varphi .$$

In view of (2.13), the phase-space volume element (3.16) in the LLRF reads

$$|\omega_n| = dVh^3(n\nu)^2 d(n\nu) \sin\theta d\theta d\varphi$$
$$= dVh^3(n\nu)^2 v_{gr}^{-1} d\nu \sin\theta d\theta d\varphi .$$

From the definition of the distribution function we thus find the number of photons in the volume dV, with frequencies ν in the range $d\nu$ and directions in the solid angle $\sin\theta d\theta d\varphi$ about (θ, φ) as seen at x^α:

$$h^3(n\nu)^2 v_{gr}^{-1} f(x^\alpha, h\nu, \theta, \varphi) dV d\nu \sin\theta d\theta d\varphi .$$

The specific radiation intensity, defined as the energy flux per unit solid angle per unit frequency range, is obtained by multiplying the last expression by $h\nu v_{gr}$ because the photons with the energy $h\nu$ propagate with the velocity v_{gr};

$$I_\nu = h^4 \nu^3 n^2 f(x^\alpha, h\nu, \theta, \varphi) . \tag{3.20}$$

If the radiation streams through the medium in an arbitrary gravitational field and is affected by the medium owing only to refraction and dispersion, Liouville's equation guarantees that the distribution function does not change along the ray. Therefore, (3.20) implies

$$\frac{I_\nu}{\nu^3 n^2} = \text{const} \tag{3.21}$$

as measured in the LLRF's along the ray. This generalizes the well-known relation $I_\nu/\nu^3 = \text{const}$ along a ray in vacuum, which is often used in the discussions of optical observations in relativistic astrophysics (e.g., in cosmology, in the studies of the optical appearance of a collapsing star, or of a star orbiting a large black hole).

As a special case, consider a refractive and dispersive medium at rest in an inertial frame in special relativity. Equation (3.21) implies

$$\nu^{-3} \frac{d}{dl}\left(\frac{I_\nu}{n^2}\right) - 3\nu^{-4} \frac{d\nu}{dl} \frac{I_\nu}{n^2} = 0 ,$$

where dl is a spatial distance corresponding to an increment dw in the parameter along the ray [see (2.27)]. Using (2.32) (with $\sigma_{\alpha\beta} = \theta = a_\alpha = 0$), we immediately obtain the classical result

$$\frac{d}{dl}\left(\frac{I_\nu}{n^2}\right) + 3 \frac{\partial n}{\partial t}\left(\frac{I_\nu}{n^2}\right) = 0 .$$

[Cf. Eq. (5.69) in Pomraning (1973), and the long derivation preceding it.]

The right-hand side of the equation of radiative transfer is usually analyzed in the LRF (see, e.g., Novikov and Thorne, 1973). In view of (3.21), in the LRF we can write the equation of radiative transfer along a given ray in the form

$$\frac{d}{dl}\left(\frac{I_\nu}{\nu^3 n^2}\right) = \text{(interaction effects with the medium)} .$$

Evaluating the left-hand side with the help of (2.32), one can directly generalize the general relativistic radiation transfer theory as described by Novikov and Thorne (1973).

IV. Radiative Transfer in Dispersive Media: Examples

1. Black-body Radiation

Consider a source emitting radiation, which has a black-body spectrum in a transparent dispersive medium near the source. In the neighbourhood of an event x_e^α, the energy density of the black-body radiation in the medium is given by (Bekefi, 1967)

$$u_{\nu_e} = \frac{8\pi h \nu_e^3 n_e^2}{\exp(h\nu_e/kT_e) - 1} \left[\frac{\partial}{\partial \nu}(n\nu)\right]_{\nu_e} ,$$

so that the specific intensity reads

$$I_{\nu_e} = \frac{2h\nu_e^3 n_e^2}{\exp(h\nu_e/kT_e) - 1} , \tag{4.1}$$

where the subscript refers to the "emission" event x_e^α, and where $n_e \equiv n(x_e^\alpha, \nu_e)$. Suppose the radiation propagates through the medium without interaction except for refraction and dispersion. Equation (3.21) then implies that an observer at rest with respect to the medium, observing the radiation at an event x_0^α, will see the spectrum

$$I_{\nu_0} = \frac{2h\nu_0^3 n_0^2}{\exp[h\nu_e(\nu_0)/kT_e] - 1} , \tag{4.2}$$

in which ν_e can be expressed by means of ν_0 by integrating (2.32) in the LRF's along the ray connecting x_e^α and x_0^α. The spectrum (4.2) will be that of black-body radiation with the temperature $T_0 = T_e \nu_0/\nu_e$ whenever the redshift ν_0/ν_e, found from (2.32), is independent of the frequency emitted. An inspection of (2.32) reveals that this, in general, will be the case if the medium is non-dispersive or, allowing for dispersion, if the refractive index is independent of time as measured in the LRF, and both the expansion and the shear of the medium vanish.

2. Radiative Transfer in a Dispersive Medium in the Friedmann Universe

Consider a homogeneous and isotropic dispersive medium at rest with respect to the fundamental observers in the Friedmann universe. Assume the refractive index to be a function of the cosmic time (t) and of the frequency. Let us focus on a fundamental observer who sees a pulse of radiation moving with the velocity $v_{gr} = \left[\frac{\partial}{\partial \nu}(n\nu)\right]^{-1}$ at time t. After an interval dt the radiation has travelled the distance $dl = v_{gr} dt$ and is passing a second fundamental observer who is moving away from the first with the velocity $v_2 = dl\,\dot{a}/a$ [$a = a(t)$ is the expansion function, the dot denotes d/dt]. Owing to the first-order Doppler effect, the second observer detects a frequency lower by $d\nu = -\nu n \dot{a} a^{-1} dl$. The frequency also changes because of the time dependence of the refractive index, namely, by $d\nu = -\nu \frac{\partial n}{\partial t} dl = -\nu \dot{n} v_{gr} dt$. Putting both effects together [and using (2.17)] we find that

$$\frac{d\nu}{dt} = -\nu\left(1 + \frac{\nu}{n}\frac{\partial n}{\partial \nu}\right)^{-1}\left(\frac{\dot{a}}{a} + \frac{\dot{n}}{n}\right). \tag{4.3}$$

Of course, the same result can be obtained from the general formula (2.32) (in which $\theta = \dot{a}/a$, $\sigma_{\alpha\beta} = a_\alpha = 0$), but the procedure above – which is a generalization of the derivation of the standard redshift formula by Peebles (1971) – is more intuitive.

Let us assume the dispersive medium to be a cold tenuous plasma with the refractive index (2.18), where $\nu_p^2 = e^2 n_{el}/m_{el}\pi$. Since the electron number density n_{el} decreases due to the expansion as a^{-3}, we can write $\nu_p^2 = Ka^{-3}$, K being a constant. Substituting this refractive index into (4.3), we arrive at a differential equation for ν as a function of a. This equation can be explicitly solved to yield

$$\nu = (c/a^2 + K/a^3)^{1/2}, \qquad c = \text{const},$$

so that

$$\nu_0 = [(a_e/a_0)^2(\nu_e^2 - \nu_{pe}^2) + \nu_{p0}^2]^{1/2}, \tag{4.4}$$

where the subscripts "0" and "e" refer to the observation and emission event. Note that, owing to the dispersion, the redshift depends on the frequency emitted. The "refractive" redshift may really occur if there is an intergalactic plasma, but it is practically negligible even for very low radio frequencies. Assuming the present number density of intergalactic electrons to be $\sim 10^{-5}$ cm^{-3}, it follows from (4.4) that

$$\nu_0 \simeq \frac{a_e}{a_0}\nu_e - \frac{1}{2}\left(\frac{a_0}{a_e}\right)^2 \frac{9 \times 10^2}{\nu_e}, \qquad [\nu] = \text{Hz},$$

the first term corresponding to the standard cosmological redshift, the second term to the "refractive" redshift. For example, $a_0/a_e = 4$ and $\nu_e = 4 \times 10^5$Hz, the "refractive" redshift is about seven orders of magnitude smaller than the standard cosmological redshift:

$$z_{\text{tot}} = (\nu_e/\nu_0) - 1 = z_{\text{cosm}} + z_{\text{refr}} \simeq [(a_0/a_e) - 1] + [4.5 \times 10^2 (a_0/a_e)^4 \nu_e^{-2}] = 3 + 7.2 \times 10^{-7}.$$

Refraction and dispersion of radiation in the intergalactic plasma certainly took place to a greater degree in earlier epochs, but then other interaction effects also have to be taken into account. Even at the present epoch, however, the dispersive properties of the intergalactic medium might reveal themselves by affecting travel times of low-frequency radiation from quasi-stellar sources (see Haddock and Sciama, 1965).

3. The Equation of Radiative Transfer in a Spherically Symmetric Case

Radio waves may be significantly affected by both dispersion and strong gravity in plasma clouds around compact objects like neutron stars and black holes. In the following we derive the explicit form of the equation of transfer in a spherically symmetric case.

Consider a spherically symmetric dispersive medium which is non-stationary in general. The medium may be the main source of the curvature of the spherically symmetric geometry (e.g. a collapsing star), or it may be regarded as test matter, moving in a given spherically symmetric background spacetime (e.g. a plasma cloud falling into a Schwarzschild black hole).

The metric in the comoving frame of the medium can be written in the form

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{4.5}$$

where Φ, Λ and R are functions of r and t. [We follow the notation of Lindquist (1966) who has given the explicit form of the general relativistic transfer equation in a spherically symmetric medium without refractive and dispersive properties.] The 4-velocity of the medium $\boldsymbol{U} = e^{-\Phi}\frac{\partial}{\partial t}$, together with the vectors $\boldsymbol{e}_1 = e^{-\Lambda}\frac{\partial}{\partial r}$, $\boldsymbol{e}_2 = R^{-1}\frac{\partial}{\partial \theta}$, and $\boldsymbol{e}_3 = (R\sin\theta)^{-1}\frac{\partial}{\partial \varphi}$ form an orthonormal tetrad at each event (a basis in the LRF). The spherical symmetry of the geometry and of the medium implies that the distribution function f depends only on four variables, which can be conveniently chosen as follows: $E=h\nu$ – the energy of a photon as measured in the LRF; $\mu=\cos\bar{\theta}$ – the cosine of the angle, measured in the LRF, between the projection of $\boldsymbol{p}$ onto the 3-space orthogonal to $\boldsymbol{U}$ (i.e. the direction of propagation) and the radial direction given by $\boldsymbol{e}_1$; and coordinates r and t. The refractive index n may be a function of only r, t, and ν. (f was chosen to depend on E, instead of ν itself, to reach exact agreement with Lindquist's notation.) Now, it can be seen easily [by calculating $\boldsymbol{p}_\perp$, normalizing it, projecting onto $\boldsymbol{e}_1$, and using the dispersion relation (2.10) and (2.11)] that

$$\mu=-\frac{e^{\Phi-\Lambda}}{n}\frac{p_1}{p_0}, \tag{4.6}$$

and, of course,

$$E=h\nu=-e^{-\Phi}p_0\,, \tag{4.7}$$

p_0 and p_1 being covariant components of $\boldsymbol{p}$ in coordinates $\{t, r, \theta, \varphi\}$.

In order to find the explicit form of the equation of transfer (3.19), we start out directly from $\boldsymbol{L}$ as given by (3.1), and assume the forms of $\boldsymbol{U}$, n and f as described above, so that $\boldsymbol{L}(f)$ reads

$$\boldsymbol{L}(f)=\frac{\partial\mathcal{H}}{\partial p_0}\frac{\partial f}{\partial t}+\frac{\partial\mathcal{H}}{\partial p_1}\frac{\partial f}{\partial r}+\frac{dE}{dw}\frac{\partial f}{\partial E}+\frac{d\mu}{dw}\frac{\partial f}{\partial\mu}.$$

Using (4.6) and (4.7), $\frac{dE}{dw}$ and $\frac{d\mu}{dw}$ can be expressed in terms of $\frac{dr}{dw}$, $\frac{dt}{dw}$, $\frac{dp_0}{dw}$ and $\frac{dp_1}{dw}$. Using then (2.11) and (2.12a) and (2.12b), all "streaming" terms in the transfer equation can be explicitly found. It is convenient to write the transfer equation in terms of the operators (cf. Misner and Sharp, 1964; Lindquist, 1966)

$$D_t\equiv e^{-\Phi}\frac{\partial}{\partial t},\quad D_r\equiv e^{-\Lambda}\frac{\partial}{\partial r},$$

and the variables

$$U\equiv D_tR\,,\quad \Gamma\equiv D_rR\,.$$

Somewhat lengthy but straightforward calculations yield the final explicit form of the equation of radiative transfer in a dispersive medium in the spherically symmetric case:

$$\begin{aligned}&nE\left(n+\nu\frac{\partial n}{\partial\nu}\right)D_tf+nE\mu D_rf\\&-nE^2\left[\mu D_r\Phi+n\mu^2D_t\Lambda+n(1-\mu^2)\frac{U}{R}\right.\\&\left.+D_tn+\nu\frac{\partial n}{\partial\nu}D_t\Phi\right]\frac{\partial f}{\partial E}\\&+nE(1-\mu^2)\left[-D_r\Phi+\frac{\Gamma}{R}+\frac{D_rn}{n}\right.\\&\left.+\mu\left(n+\nu\frac{\partial n}{\partial\nu}\right)\left(\frac{U}{R}-D_t\Lambda\right)\right]\frac{\partial f}{\partial\mu}=g\,.\end{aligned} \tag{4.8}$$

Provided that $n=1$, this equation reduces to the transfer equation as given by Lindquist [1966, Eq. (3.7)], but even if $n=1$ from the very beginning, our derivation differs from that of Lindquist. If the medium is non-dispersive, we may start out from $\boldsymbol{L}$ in the form (3.3) and apply Lindquist's procedure directly. (In the non-dispersive case, we can actually guess the explicit form (4.8) from Lindquist's result, if appropriate identifications, based on the existence of the optical metric, are made.)

Assuming the medium to be static and imposing the condition $R(r)=r$, we can simplify Eq. (4.8) to the following form:

$$\begin{aligned}&e^{-\Lambda}nE\left[\mu\frac{\partial f}{\partial r}+\left(\frac{1-\mu^2}{r}+\frac{1-\mu^2}{n}\frac{dn}{dr}\right)\frac{\partial f}{\partial\mu}\right]\\&-e^{-\Lambda}nE\frac{d\Phi}{dr}\left[(1-\mu^2)\frac{\partial f}{\partial\mu}+\mu E\frac{\partial f}{\partial E}\right]=g\,.\end{aligned} \tag{4.9}$$

Except for the multiplication of the terms by n, the only difference between (4.9) and the transfer equation for a medium without refractive properties is the term $e^{-\Lambda}nE\frac{1-\mu^2}{n}\frac{dn}{dr}\frac{\partial f}{\partial\mu}$. Provided that the identification $n\leftrightarrow e^{-\Phi}$ is made, the last term is formally the same as $-e^{-\Lambda}nE(1-\mu^2)\frac{d\Phi}{dr}\frac{\partial f}{\partial\mu}$, i.e. a term which can be identified with the gravitational deflection of radiation. Thus, also in the transfer equation a well-known equivalence[3] between the gravitational field and the optical medium is exhibited.

V. Concluding Remarks

As demonstrated, there appear to be no obstacles in extending the general relativistic radiation transfer theory to media with refractive and dispersive effects. In general, of course, we should consider anisotropic media such as, for example, plasma in a magnetic field, and we should also incorporate polarizational properties of radiation into the equation of transfer in a manner similar to that used recently by Anile and Breuer (1974) for non-refractive and non-dispersive media in general relativity.

The methods of solving the relativistic transfer equation will, in most of the tractable cases, not differ from known classical methods. In particular, they will be similar in spherically symmetric cases.

As indicated in the Introduction, one can encounter astrophysically plausible situations in which refraction and dispersion as well as the curvature of spacetime can play an important role. However, there are problems which do not involve curvature at all, but nevertheless can be tackled to an advantage by using the generally covariant form of the transfer equation. For example,

[3]) For a recent discussion of this equivalence, see de Felice (1971)

Castor (1972) employed the method of Lindquist [yielding Eq. (4.8) with $n \equiv 1$] in order to study radiative transfer in spherically symmetric flows with the in-inclusion of special-relativistic corrections. Castor's analysis, motivated by high-velocity flows in quasi-stellar objects, should be easily extendable to flows of dispersive and reftactive media by using the procedure leading to Eq. (4.8).

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